UDC 532.529

CONTINUAL MECHANICS OF MONODISPERSE SUSPENSIONS. ON PROPERTIES OF SPHERICAL DIPOLE SUSPENSIONS IN AN EXTERNAL FIELD

PMM Vol. 38, №2, 1974, pp. 301-311
Iu. A. BUEVICH
(Moscow)
(Received June 28, 1973)

A system of equations of conservation which determines the macroscopic motion of a moderately concentrated suspension of solid spherical particles subjected to moments exerted by an external field is derived. Rheological parameters which define the quasi-stationary behavior of such suspension in the presence of an external field whose intensity is independent of the motion of suspension are determined.

Suspensions of "loaded" particles whose center of gravity is not at their geometric center in a gravitational or centrifugal field and suspensions of magnetized particles or particles carrying electric dipoles in an electromagnetic field, are examples of disperse systems of particles with dipole moments. Investigations of dilute suspensions of this kind (see, e.g., [1 – 6]) show that the action of external force couples on suspended particles considerably affects the rheological properties of suspension and leads to the emergence of qualitatively new effects, in particular, the appearance in the stream of a nonzero antisymmetric tensor component. This component depends on the orientation of vectors of the external field intensity and of the curl of the suspension mean velocity with respect to the relative moduli of that curl and of external couple. The latter violates the Newtonian properties of the stream and results in the formation of non-Newtonian properties of suspension. These theoretical conclusions are supported by experimental data on viscosimetry of suspensions of magnetized particles, which were recently widely publicized [7 – 8]. Of considerable interest is the extension of results of investigations [1 – 6] to highly concentrated suspension of dipole particles in streams in which the antisymmetric and non-Newtonian properties are particularly pronounced. The continuous suspension model developed in [9, 10] can be used for this purpose. Such extension of the theory of dilute suspensions of spherical dipoles [1, 2] to moderately concentrated suspensions is proposed below.

1. Let us consider a suspension of solid spheres of radius a with a dipole moment $\mathbf{D} = D\mathbf{T}$, where $\mathbf{T}$ is a unit vector frozen into a particle and the quantity D is the same for all particles. The suspension is in an external field of intensity $\mathbf{g}$ which exerts on particles the moment

$$\mathbf{L} = \mathbf{D} \times \mathbf{g} = D(\mathbf{T} \times \mathbf{g}) \tag{1.1}$$

The suspension is in an external mass field with potential Φ. If loaded spheres whose center of gravity lies at distance $\mathbf{x}$ from their geometric centers are considered, then

$$\mathbf{T} = \mathbf{x} / x, \quad D = {}^4/_3\, \pi a^3 d_1 x, \quad \mathbf{g} = -\nabla\Phi \tag{1.2}$$

where d_1 is the mean density of the particle material and vector g may be considered to be a quantity known *a priori* and independent of the suspension state. In particular, g may represent the acceleration of gravity.

The situation becomes much more complicated, if particles with electric (magnetic) dipole moments are considered, since then vector g in (1.1) defines the local intensity of the electric (magnetic) field in the vicinity of a particle, which generally differs from the mean intensity and depends on the configuration of adjacent particles. Moreover, the mean intensity represents the solution of Maxwell equations in the region occupied by the suspension and, to a great extent, depends on the distribution of effective permittivity and magnetic permeability of suspension in that region. These quantities are determined not only by the related permittivity and permeability of the particle material and the continuous suspension phase, but also by the mean volume concentration and orientation of particles. Thus in a material medium conventional equations of conservation of mass, momentum, and moment of momentum of suspension phases are linked to the system of Maxwell equations, and this brings forth the additional very complicated problem of derivation of the latter equations in their explicit form. We point out that the indicated here "physical" nonlinearity and the fundamental difference between system of loaded particles and of particles with electric or magnetic dipole moments were not mentioned in [1 – 6], which led to the erroneous conclusion in [2] about the complete analogy between these systems. As shown by experiments [7, 8], the effect of suspension of the external electromagnetic field can be considerable, as for instance, in the case of ferromagnetic particles.

That effect may to a certain extent be neglected only if two conditions are satisfied. First, permittivity and magnetic permeability of the two phases must differ only slightly from their values in vacuum. Second, the dipole moment of a unit volume of suspension which determines the effective vector of dielectric polarization (intensity of magnetization) of the suspension and is produced by dipole moments of its particles, must be small in comparison with the external field intensity. The vector in (1.1) can then be approximately considered as a quantity independent of the state and behavior of suspension and to be definable by solving Maxwell equations for vacuum conditions.

We also neglect the effect of Brownian rotary motion of suspended particles. The above conditions , and also the conditions of the quasi-stationary state of suspension defined below, are presented in analytic form at the end of this paper.

We consider here the assumptions made in [9, 10] in the derivation of the suspension model to be valid. In particular we assume the Reynolds number which defines the flow around individual particles to be small and the space distribution of particles to be random. The effect of particle disjointness, which is admissible if the concentration of particles is not too great. These assumptions and ensuing limitations were considered in detail in [10]. The linear scale of variables which define an observable macroscopic flow of suspension is assumed to be considerable in comparison with the scale of suspension microstructure and to be of the order of the average distance between adjacent suspended particles. The latter is the necessary condition for the averaging over a small volume of suspension to be valid and for considering its phases as two interpenetrating and interacting continua [9].

2. Repeating the reasoning of [9], after averaging over the volume we obtain the equations of conservation of mass, momentum and moment of momentum of phases, which differ from the equations in [9] only by the additional term

$$n\mathbf{l} = \frac{1}{b}\sum_j \mathbf{L}^{(j)} = D\left(\frac{1}{b}\sum_j \mathbf{T}^{(j)}\right) \times \mathbf{g} = nD\,(\boldsymbol{\tau} \times \mathbf{g}) \tag{2.1}$$

which defines the total moment exerted by field $\mathbf{g}$ on particles in a unit of volume and appears in the right-hand part of the equation of conservation of the moment of momentum of the disperse phase. Summation in (2.1) is carried out over particles in the small physical volume b containing a number of particles sufficient for averaging. The subscript (j) denotes the particle item number, n is the denumerable concentration of particles, and $\mathbf{l}$ and $\boldsymbol{\tau}$ represent the values of the external moment $\mathbf{L}$ and vector $\mathbf{T}$ averaged over a great number of particles under identical conditions. We point out that the absolute value of vector $\boldsymbol{\tau}$ is smaller than unity, reaching unity only in the limit case of identical orientation of all particles.

The mean force and moment of interphase interaction, the symmetric part of the mean stress tensor, and the pseudo-tensor of moment stresses, which appear in the equation of conservation of the moment of momentum of the continuous phase, are expressed in terms of mean stresses at the surface of an individual particle exactly in the same manner as in [9]. The only difference in this case is that it contains the antisymmetric component of the mean stress tensor

$$\boldsymbol{\sigma}^{(a)} \quad {}^1/_2 n\boldsymbol{\varepsilon}\,\mathbf{l} \tag{2.2}$$

where ε is the alternating antisymmetric Levi-Civita tensor. Formula (2.2) is analogous to the formula for dilute suspensions [2 – 4]. The sign in (2.2) is chosen in accordance with the condition that the divergence of $\sigma^{(a)}$ appearing in equations is calculated by differentiating with respect to the second subscript.

We use the scheme developed in [10] for calculating mean stresses at the surface of an individual particle. We carry out the averaging over the conditional distribution function of the ensemble of all particles, except some isolated (sample) particle whose center is at point $\mathbf{r}$ and orientation vector is $\mathbf{T}$. Without entering into details, we point out that the only difference between the ensemble distribution functions used here and similar functions in [10] is in the appearance of orientation vectors of particles as additional arguments of these functions. As the result, we obtain, as previously, the problem of flow of a fictitious homogeneous medium around a sample particle rotating at angular velocity $\boldsymbol{\Lambda}^*$ dependent on $\mathbf{T}$. Taking in addition into account that by stipulation the linear scale of n and $\mathbf{l}$ considerably exceeds radius a, which determines the scale of perturbations induced by the sample particle in the flow of the fictitius medium, we conclude that in the investigation of flow around the sample particle it is necessary, in accordance with the general method of [10], to disregard the quantity (2.2). Hence the properties of the fictitious medium are in this case identical to those of the medium considered in [10], and the problem of flow around a sample particle differs from that in [10] only by that instead of the true mean angular velocity $\boldsymbol{\lambda}$ the velocity $\boldsymbol{\Lambda}^*$ appears in it. It is obvious that

$$\boldsymbol{\lambda} = \int \boldsymbol{\Lambda}^*(\mathbf{T})\,\varphi(\mathbf{T})\,d\mathbf{T} \tag{2.3}$$

where $\varphi(\mathbf{T})$ is the distribution function of directions $\mathbf{T}$ normalized with respect to unity.

For the unknown stresses at the particle surface we obtain the same expressions as in

[10] in which Λ^* is substituted for λ so that for the final determination of mean stresses at the particle surface it is necessary to carry out an additional averaging over $\varphi(\mathbf{T})$. The mean interphase force, the symmetric components of suspension mean stresses, and the mean moment stresses are generally independent of λ and Λ^* and, consequently, are the same as those calculated in [10]. The mean hydrodynamic moment $\mathbf{M}^*$ exerted by the surrounding medium on the sample particle is determined by the relationship

$$\mathbf{M}^* = 8\pi a^3 \mu_0 (M^{(1)} \mathbf{v} - M^{(2)} \Lambda^*), \qquad \mathbf{v} = {}^1/_2 \operatorname{rot} \mathbf{c} \tag{2.4}$$

where $\mathbf{c}$ is the mean velocity of suspension, and $M^{(1)}$ and $M^{(2)}$, which depend only on the volume concentration ρ of particles and for $\rho \to 0$ reduce to unity, were determined in [10]. Note that (2.4) is valid when the characterisitic frequency ω of flow is considerably lower than frequency $\omega_0 = \mu_0(d_0 a^2)^{-1}$, where d_0 and μ_0 are the density and viscosity of the continuous phase. Averaging $\mathbf{M}^*$ in (2.4) over directions $\mathbf{T}$, we obtain for the mean interphase moment the previous expression.

Thus on the basis of results in [9, 10] for $\omega \ll \omega_0$ for the macroscopic motion of the considered suspension we have the following equations:

$$\partial\varepsilon / \partial t + \nabla(\varepsilon \mathbf{v}) = 0, \quad \partial\rho / \partial t + \nabla(\rho \mathbf{w}) = 0, \quad \varepsilon = 1 - \rho \tag{2.5}$$

$$d_0\varepsilon(\partial / \partial t + \mathbf{v}\nabla)\mathbf{v} = -\nabla p + 2\nabla(\mu \mathbf{e}) + \nabla \sigma^{(a)} - \mathbf{f} - d_0 \nabla\Phi \tag{2.6}$$

$$d_1\rho(\partial / \partial t + \mathbf{w}\nabla)\mathbf{w} = \mathbf{f} - (d_1 - d_0)\rho\nabla\Phi$$

$$d_0\varepsilon(\partial / \partial t + \mathbf{v}\nabla)\mathbf{K}_0 = 2\nabla(\eta \mathbf{y}) - {}^1/_5 a^2 \nabla(\varepsilon \mathbf{f}) + \mathbf{h} - \mathbf{m} \tag{2.7}$$

$$d_1\rho(\partial / \partial t + \mathbf{w}\nabla)\mathbf{K}_1 = \mathbf{m} + n\mathbf{l}, \quad \mathbf{h} = \| \varepsilon_{ijk}\sigma_{ik}^{(a)} \|, \quad \mathbf{K}_1 = \mathbf{K}_1(\lambda)$$

where $\mathbf{v}$ and $\mathbf{w}$ are the mean velocities of the continuous and the dispersed phases, respectively, p is the mean pressure in the continuous phase, and $\mathbf{K}_0$ and $\mathbf{K}_1$ are the mean internal moments of momentum of phases per unit of their volume, as determined in [9]. Specific formulas for viscosity μ, moment viscosity η and for the mean interphase force $\mathbf{f}$ and moment $\mathbf{m}$ per unit volume of suspension appear in [10]. Tensors $\mathbf{e}$ and $\mathbf{y}$ are determined by the equalities

$$\mathbf{e} = \frac{1}{2}\left\| \frac{\partial c_i}{\partial r_j} + \frac{\partial c_j}{\partial r_i} \right\|, \quad \mathbf{y} = \frac{1}{2}\left\| \frac{\partial(\operatorname{rot}\mathbf{c})_i}{\partial r_j} + \frac{\partial(\operatorname{rot}\mathbf{c})_j}{\partial r_i} \right\| \tag{2.8}$$

Note that the divergence of tensors in (2.6) and (2.7) are computed by differentiation with respect to the second subscript.

The problem of determination of rheological equations for the state of suspension is, thus, reduced to the calculation of $\mathbf{l}$ in (2.1), after which tensor $\sigma^{(a)}$ can be readily determined with the use of (2.2).

3. Assuming that the characteristic frequency ω is low in comparison with $\omega_1 = \mu_0(d_1 a^2)^{-1}$, we can neglect the inertia of the sample particle and write the equation of its rotation in the form

$$\mathbf{M}^* + D(\mathbf{T} \times \mathbf{g}) = 0 \tag{3.1}$$

thus obtaining for the angular velocity of the sample particle in (2.4) and (3.1) the expression

$$\Lambda^* = \frac{M^{(1)}}{M^{(2)}}\mathbf{v} + \frac{D}{8\pi a^3 \mu_0 M^{(2)}}(\mathbf{T} \times \mathbf{g}) \tag{3.2}$$

Using (3.2) in the obvious equation

$$dT / dt = \Lambda^* \times T \tag{3.3}$$

which describes the rotation of the frozen-in vector T, we obtain equation

$$dT / dt = \alpha [v^\circ \times T + \beta (g^\circ - T(g^\circ T))] \tag{3.4}$$

which is of the same form as the equation analyzed in [1, 2]. Here v° and g° are unit vectors in the directions of v and g, and the following parameters have been introduced:

$$\alpha = \frac{vM^{(1)}}{M^{(3)}}, \quad \beta = \frac{Dg}{8\pi a^3 \mu_0 v M^{(1)}} \tag{3.5}$$

The solution of Eq.(3.4) determines the trajectories of vectors T and by the same token the distribution function $\varphi(T)$.

In the general case the solution of (3.4) presents considerable difficulties. Here, following [1, 2], we consider only the asymptotic state which is formally attainable for $t \to \infty$, hence the following results are, strictly speaking, applicable only to the analysis of quasi-stationary flows. The characteristic relaxation time T is, in accordance with (3.4), equal α^{-1}. Hence the condition of quasi-stationary state for the characteristic frequency can be written in the form $\omega \ll \alpha$.

The analysis by Hall and Busenberg [1] and Brenner [2] shows that there are two qualitatively different kinds of asymptotic solutions of equations of the form (3.4) whose realization depends on parameter β in (3.5) and angle γ between the directions of v° and g°. If $\beta \geqslant 1$ or lies in the interval [0, 1], but γ is not equal $^1/_2\pi$, then for $t \to \infty$ vector T assumes an entirely determined position τ which from thereon is independent of time and defines the final orientation of particles. As was shown in [1] with the use of the Poincare-Bendixon stability theorem, this final orientation is independent of the initial position of vector T is unique and stable with respect to perturbations of τ of any arbitrary amplitude and direction. Vector τ is determined by the solution of the stationary analog of Eq. (3.4), which is of the form

$$\tau = v^\circ \cos\psi_1 \sin^2\psi_2 + g^\circ \frac{\sin\psi_1 \cos\psi_2}{\sin\gamma} + (v^\circ \times g^\circ) \frac{\sin\psi_1 \sin\psi_2}{\sin\gamma} \tag{3.6}$$

where angles ψ_1 and ψ_2 are defined by

$$\sin\psi_1 = \{^1/_2 (1 + \beta^2) - [^1/_4 (1 + \beta^2)^2 - \beta^2 \sin^2\gamma]^{1/2}\}^{1/2} \tag{3.7}$$
$$\sin\psi_2 = (\beta \sin\gamma)^{-1} \sin\psi_1, \quad \sin\gamma = + [1 - (v^\circ g^\circ)^2]^{1/2}$$

The choice of sign for $\sin\gamma$ is self evident; the sign of the expression for $\sin\psi_1$ in (3.7) is chosen so that angle ψ_1 between vectors τ and v° lies in the same quadrant as γ.

It is evident that the asymptotic distribution function $\varphi(T)$ is in this case a delta-function, hence the averaging over $\varphi(T)$ reduces in fact to the substitution of τ for T

For the asymptotic angular velocity of rotation of a particle from (2.3) and (3.3) we obtain

$$\Lambda^* = \lambda = \lambda\tau \tag{3.8}$$

i.e. the particles rotate around axes parallel to τ. Hence from (2.4) and (3.1) for the external moment L^* exerted on the sample particle by the external field we obtain

$$L^* = l = D(\tau \times g) = 8\pi a^3 \mu_0 (M^{(1)} \lambda\tau - M^{(2)} v) \tag{3.9}$$

Using the definition of angle ψ_1 Eq. (3. 8), and the scalar product of (3. 9) by $\boldsymbol{\tau}$, we further obtain

$$\lambda = (M^{(1)} / M^{(2)})\, \boldsymbol{\nu\tau} = \alpha \cos \psi_1 \tag{3.10}$$

It follows from this and (2. 1), (2. 2), (3. 8) and (3. 9) that

$$\begin{aligned} n\mathbf{l} &= 6\rho M^{(1)}\mu_0 \nu\, (\boldsymbol{\tau} \cos \psi_1 - \boldsymbol{\nu}^\circ) \\ \boldsymbol{\sigma}^{(a)} &= 3\rho M^{(1)}\mu_0 \nu \varepsilon\, (\boldsymbol{\tau} \cos \psi_1 - \boldsymbol{\nu}^\circ) \end{aligned} \tag{3.11}$$

which finally closes the system of equations of conservation (2. 5) – (2. 7). The denumerable concentration of particles n is defined in (3. 11) in terms of volume concentration ρ. When $M^{(1)} \to 1$, a similar result is obtained for dilute suspensions from (3. 11) for $\rho \to 0$ [1, 2]. Note that the sign of $\boldsymbol{\sigma}^{(a)}$ in (3. 11) is opposite to its sign in [1, 2], which is due to the difference in the definition of divergence of this tensor in the equation of conservation of momentum of the continuous phase.

The dipole moment of a unit of suspension volume is expressed in terms of the sum of moments of particles, i. e.

$$\mathbf{d} = nD\boldsymbol{\tau} = (3\rho D / 4\pi a^3)\, \boldsymbol{\tau} \tag{3.12}$$

Since, as implied by (3. 6), the components of vector $\boldsymbol{\tau}$ are normal to the direction of the external field $\mathbf{g}^\circ$, the suspension is anisotropic in the sense that its vector of dielectric polarization or magnetization is proportional to $\mathbf{d}$ defined by (3. 12) and lies at an angle to $\mathbf{g}^\circ$.

In the particular case of $\gamma = {}^1/_2\pi$ and $0 \leqslant \beta < 1$ the sample particle rotates about a certain axis lying in a plane normal to $\mathbf{g}^\circ$, and the asymptotic orientation of the axis in the plane is determined by the particle initial orientation [1, 2]. In that case vector $\mathbf{T}$ performs a periodic rotation, lying at any instant of time on the generatrix of some right circular cone whose vertex angle depends on the initial direction of $\mathbf{T}$. The form of the distribution function $\varphi\,(\mathbf{T})$ also depends on the initial distribution of particle orientation. More or less plausible assumptions can be made about that distribution.

The first hypothesis of this kind was the assumption of a homogeneous initial distribution of orientations made in [1]. However such distribution cannot be actually obtained, since it does not satisfy the stationary Liouville equation [3]. It was proposed in that paper to use one of the solutions of that equation, without however any substantiation that that particular solution is realized.

An attempt was made in [2] at obtaining a quasi-stationary distribution $\varphi\,(\mathbf{T})$ for $\gamma = {}^1/_2\pi$ from the analysis of distributions corresponding to $\gamma = {}^1/_2\pi + \delta$ with $\delta \to 0$. For small δ from (3. 7) we have

$$\sin \psi_1 \approx \beta, \qquad \sin \psi_2 \approx 1 \tag{3.13}$$

hence in terms of $n\mathbf{l}$ and $\boldsymbol{\sigma}^{(a)}$ the indicated limits are expressed by

$$\begin{aligned} n\mathbf{l} &= -6\rho M^{(1)}\mu_0 \nu\beta\, [\beta \boldsymbol{\nu}^\circ \pm (1 - \beta^2)^{1/2}\, (\boldsymbol{\nu}^\circ \times \mathbf{g}^\circ)] \\ \boldsymbol{\sigma}^{(a)} &= -3\rho M^{(1)}\mu_0 \nu\beta\varepsilon\, [\beta \boldsymbol{\nu}^\circ \pm (1 - \beta^2)^{1/2}\, (\boldsymbol{\nu}^\circ \times \mathbf{g}^\circ)] \end{aligned} \tag{3.14}$$

where the upper and lower signs relate to positive and negative δ. It will be seen that these limits are substantially different (particularly for small β). Assuming that the mean of these limits correspond to $\gamma = {}^1/_2\pi$, we obtain

$$n\mathbf{l} = -6\rho M^{(1)}\mu_0 \beta^2 \boldsymbol{\nu}, \quad \boldsymbol{\sigma}^{(a)} = -3\rho M^{(1)}\mu_0 \beta^2 \boldsymbol{\varepsilon\nu} \tag{3.15}$$

The derivation of (3.15) is based on the concept that in real situations parameter δ represents a random function which defines the scatter of angles γ in the vicinity of various particles and is the result of weak local inhomogeneities of the stream, and also on the fact that vector $\mathbf{T}$ is in position $\boldsymbol{\tau}$, as defined by (3.6), no matter how small is angle δ.

However for small $|\delta|$ the stability of stationary orientation is disrupted in the sense that a comparatively small perturbation is sufficient for a forced transition of γ through $^1/_2\pi$ accompanied by a substantial change of the pattern of the particle behavior. In particular, however small the effect of rotary Brownian motion on the formation of the rheological properties of the stream with $\gamma \neq {}^1/_2\pi$ (see e.g., [6, 11]), that effect becomes decisive in the region of $\gamma \approx {}^1/_2\pi$ [12]. Physical considerations make it obvious that in that region Brownian motion must lead to some completely stabilized distribution independent of the initial state of suspension.

Since the analysis of Brownian motion is outside the scope of the problem considered here, we present only the computation of quantities (2.1) and (2.2) at the limit of a vanishing weak Brownian motion for $\gamma = {}^1/_2\pi$. This is readily carried out by using the limit distribution function $\varphi(\mathbf{T})$ derived by Hinch and Leal [12] for dilute suspensions. Using the method of [12] it can be readily shown that the form of that function remains unchanged in the case of suspensions of moderate concentration considered here. The final formulas are of the form

$$n\mathbf{l} = -6\rho M^{(1)}\mu_0 F(\beta)\,\mathbf{v}, \qquad \boldsymbol{\sigma}^{(a)} = -3\rho M^{(1)}\mu_0 F(\beta)\,\boldsymbol{\varepsilon}\mathbf{v} \qquad (3.16)$$

$$F(\beta) = 1 - \frac{1-\beta^2}{\sqrt{3}}\,\frac{\ln[(\sqrt{2+\beta^2}+\beta\sqrt{3})(\sqrt{2+\beta^2}-\beta\sqrt{3})^{-1}]}{\ln[(\sqrt{2+\beta^2}+\beta)(\sqrt{2+\beta^2}-\beta)^{-1}]} \qquad (3.17)$$

For $\beta \lesssim 1$ the quantities defined by (3.16) substantially differ from those in (3.14), owing to previously noted effect of Brownian motion on the change of the rotation pattern of particles. The condition for realizing (3.14) can be obtained in exactly the same manner as in [12]. We have

$$\frac{D_{Br}}{\nu M^{(1)}} \ll |\delta|(1-\beta)^{-1/2}, \qquad D_{Br} = \frac{kT}{8\pi a^3\mu_0 M^{(2)}} \qquad (3.18)$$

where D_{Br} is the coefficient of rotational Brownian diffusion defined in (3.18) in terms of temperature in energy units in conformity with Einstein's formula. Formulas (3.16) remain valid for the reversed inequality (3.18). Formulas for intermediate values of $|\delta|$ can be obtained in principle by applying to the equation for $d\mathbf{T}/dt$ which contains a diffusion term of the method of matching asymptotic expansions.

Thus the rheological parameters of a suspension related to the presence of dipole interaction with the external field suffer at transition through $\gamma = {}^1/_2\pi$ and abrupt change in the region

$$\Delta\gamma \sim |\delta| \sim D_{Br}(\nu M^{(1)})^{-1}(1-\beta)^{1/2} \qquad (3.19)$$

From the point of view of the theory in which Brownian motions are not taken into consideration such abrupt change of parameters is taken as a disturbance of the continuity of their dependence on γ, i.e. on the orientation of flow relative to the external field. A similar conclusion with respect to dilute suspensions was reached in [2, 11].

We also point out that for concentrated suspensions the limitation (3.18) is less stringent than for diluted ones, since $M^{(1)}$ and $M^{(2)}$ are rapidly increasing functions of concentration [10].

4. The equations of conservation (2.5) – (2.7) together with the results presented in [10] and in Sect. 3 above make it possible to investigate flows of the simplest kind including viscosimetric ones, as was done in [2] for dilute suspensions. As an example we consider here only a Couette flow with constant concentration of suspension. For the purpose of this Sect. it is sufficient to consider the particular case of such flow in which the phase velocities $\mathbf{v}$ and $\mathbf{w}$ are equal to the velocity of suspension specified in the form $\mathbf{c} = Gx\mathbf{y}^\circ$, where G is the rate of shear, and assume

$$\mathbf{g}^\circ = \mathbf{x}^\circ \sin\gamma\cos\theta + \mathbf{y}^\circ \sin\gamma\sin\theta + \mathbf{z}^\circ\cos\gamma \tag{4.1}$$

where $\mathbf{x}^\circ$, $\mathbf{y}^\circ$ and $\mathbf{z}^\circ$ are unit vectors of the coordinate axes and θ is a certain angle which determines the direction of projection of $\mathbf{g}^\circ$ in the plane (x, y). It can be readily shown with the use of equations of conservation derived in Sect. 2 that such flow can be obtained either when the densities of particles and fluid are equal or in the absence of an external mass field. Taking into account that $\mathbf{v} = {}^1/_2 G\mathbf{z}^\circ$, after computations based on formulas derived in Sects. 2 and 3 for the components of the flow mean stress tensor we obtain the following expressions:

$$\begin{aligned}
&\sigma_{xx}' = \sigma_{yy}' = \sigma_{zz}' = 0 \qquad (\sigma_{ij}' = \sigma_{ij} + p\delta_{ij}) \\
&\sigma_{xy}' = \mu_0 G\,(1 + {}^5/_2\rho S - {}^3/_2\rho M \sin^2\psi_1) \\
&\sigma_{yx}' = \mu_0 G\,(1 + {}^5/_2\rho S + {}^3/_2\rho M \sin^2\psi_1) \\
&\sigma_{xz}' = -\sigma_{zx}' = -{}^3/_2\rho M\mu_0 G \sin\psi_1\cos\psi_1\sin(\psi_2 + \theta) \\
&\sigma_{yz}' = -\sigma_{zy}' = {}^3/_2\rho M\mu_0 G \sin\psi_1\cos\psi_1\cos(\psi_2 + \theta)
\end{aligned} \tag{4.2}$$

where $M = M^{(1)}$ and function $S = S(\rho)$ is defined by

$$\mu = \mu_0\,(1 + {}^5/_2\rho S) \tag{4.3}$$

with μ representing the effective viscosity of suspension in the absence of an external field or dipole moments of particles, and determined in [10]. For $\rho = 0$ functions $M(\rho)$ and $S(\rho)$ are equal unity and rapidly increase with ρ. The Couette flow viscosity μ' which is determined experimentally can be defined as the ratio of the tangential component of force in the direction $\mathbf{y}^\circ$ which acts on a small unit area of plane (y, z), i.e. of parameter σ_{yx}', to the shear rate G. It is more convenient to consider the reduced viscosity

$$[\mu] = \frac{\mu' - \mu_0}{\mu_0\rho} \tag{4.4}$$

In the particular cases, when the external field is parallel ($\gamma = 0$ or π) or normal ($\gamma = {}^1/_2\pi$) to the curl vector from (4.4) we have, respectively,

$$[\mu] = {}^5/_2 S, \quad [\mu] = {}^5/_2 S + {}^3/_2\beta^2 M \tag{4.5}$$

Similarly for $\beta = 0.1$ or ∞, we have, respectively,

$$\begin{aligned}
&[\mu] = {}^5/_2 S, \quad [\mu] = {}^5/_2 S + {}^3/_2 M\,(1 - |\cos\gamma|) \\
&[\mu] = {}^5/_2 S + {}^3/_2 M \sin^2\gamma
\end{aligned} \tag{4.6}$$

For dilute suspensions formulas (4.4) – (4.6) coincide with those derived in [2]. Other characteristics of the Couette flow, which are important for equations of conservation are not difficult to obtain. In particular tensor γ in (2.8) is identically zero, and moment stresses are absent in this flow.

Thus it follows from the analysis of this particular case that a suspension of dipole particles in an external field is in two respects non-Newtonian. First, the effective viscosity coefficients depend on the rate of shear which appears in the definition of parameter β in (3.5). Second, the tensor proportionality between mean stresses and rates of shear is disturbed, resulting in the appearance in (4.2) of stresses normal to the plane of flow. The suspension is not only an anisotropic dielectric or magnet, but also an anisotropic body in the hydrodynamic sense. These two conclusions are obviously of a general character.

Since M increases with increasing ρ faster then S (see [10]), the relative importance for non-Newtonian and antisymmetric properties increases with increasing concentration of the suspension.

We note that Batchelor, when considering an artificial example of suspension [4] whose particles are subjected to an external force couple independent of particle motion, concluded that such suspension is a "quasi-Newtonian" medium in the sense that mean stresses are expressed in terms of velocity derivatives with respect to coordinates by the linear tensor relationship

$$\sigma_{ij}' = \mu_{ijkm} \partial v_k / \partial r_m \tag{4.7}$$

which contains the fourth rank tensor of viscosity coefficients. It is evident that for real flows with dipole interaction between particles and the external field this conclusion is false, since not only the viscosity coefficients themselves depend on velocity derivatives but, as follows from (3.6) and (3.11), tensor σ' has a component which is proportional to εg^c.

5. In the foregoing we made use of assumptions on which the concept of continuous mechanics of suspension is based in [9, 10], where these assumptions are discussed in detail and briefly mentioned in Sect. 1 above. Here we consider only new assumptions specific to this work.

First of all, we note that the condition of independence of the external electric or magnetic field from the state of suspension requires that the related permittivity and magnetic permeability differ only slightly from those in vacuum and that in addition the inequality

$$4\pi d = 3\rho D a^{-3} \ll g \tag{5.1}$$

which confirms the low polarization (magnetization) of suspension, produced by the dipole moment (3.12) in comparison with the field intensity, be satisfied.

The condition of weak effect of Brownian motion means that the reciprocal of the Péclet number P_{Br} for rotational Brownian diffusion is small in comparison with unity. This condition may be written as

$$\frac{1}{P_{Br}} = \frac{D_{Br}}{\gamma M^{(1)}} = \frac{kT}{8\pi M^{(1)} M^{(2)} a^3 \mu_0 \gamma} \ll 1 \tag{5.2}$$

Note that this condition is violated when γ is fairly close to $^1/_2\pi$ for arbitrary small diffusion coefficient D_{Br} (see estimate in (3.18)). Condition (5.2) imposes the lower limit on the admissible radius a of suspension particles.

The results obtained here are valid for flows whose characteristic frequency ω is not excessively high. We have three related conditions of quasi-stationarity

$$\omega \ll \omega_0 = \frac{\mu_0}{d_0 a^2}, \quad \omega \ll \omega_1 = \frac{\mu_0}{d_1 a^2}, \quad \omega \ll \frac{\nu M^{(1)}}{M^{(2)}} \qquad (5.3)$$

Considering that smallness of the Reynolds number of the flow around particles implies the fulfilment of inequalities

$$a d_0 u < \mu_0, \quad a^2 d_0 \lambda < \mu_0 \qquad (u = v - w) \qquad (5.4)$$

we conclude that in real situations the first two conditions of (5.3) are usually satisfied and that the most restrictive is the third condition of quasi-stationarity. Unlike (5.2), conditions (5.3) and (5.4) impose the upper limit on the particle radius.

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Translated by J. J. D.